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Sequences of Integers and Ergodic Transformations

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We consider a σ-finite, non-atomic measure space $(X, \mathscr{B}, m)$ with a 1–1, onto measure preserving transformation T defined on it. We shall say that T is ergodic in case $TA = A$ implies $m(A) = 0$ or $m(X - A) = 0$. A sequence of integers $\{n_i\}$ is a weakly wandering sequence for the transformation T if there exists a set A of positive measure such that $T^{n_i}A \cap T^{n_j}A = \varnothing$ for $i \neq j$, and $\{n_i\}$ is an exhaustive weakly wandering sequence in case $X = \bigcup_{i=1} T^{n_i}A$ (disj). We shall also refer to the set A as a weakly wandering or an exhaustive weakly wandering set under the sequence $\{n_i\}$, respectively.

Weakly wandering sets and sequences were introduced in [8] and were shown to be useful in the study of finite invariant measures for measurable and non-singular transformations. These sequences were utilized in the study of ergodic measure preserving transformations defined on an infinite measure space (see [5, 6]). For non-singular transformations Kamae [10] characterized non-negative weakly wandering sequences. Subsequently, it was noticed that exhaustive weakly wandering sets and sequences were more effective in describing properties of ergodic measure preserving transformations defined on an infinite measure space. However, the only concrete example of an exhaustive weakly wandering sequence that was known so far was the example constructed in [9]. In this article we shall construct a class of ergodic measure preserving transformations defined on an infinite measure space that generalizes this example. We shall exhibit a large collection of exhaustive weakly wandering sequences. Furthermore, we show a close connection between exhaustive weakly wandering sequences of this class and sequences that partition the set of integers into direct summands.

Let $\mathbb{V} = \{v\}$ and $\mathbb{W} = \{w\}$ be two infinite subsets of the set of non-negative integers $\mathbb{N} = \{0, 1, 2, \ldots\}$ such that every $n \in \mathbb{N}$ can be written uniquely as $n = v + w$ with $v \in \mathbb{V}$ and $w \in \mathbb{W}$. We say that $\mathbb{N}$ is the direct sum of $\mathbb{V}$ and $\mathbb{W}$ and write $\mathbb{N} = \mathbb{V} \oplus \mathbb{W}$.

In [9] an example of an ergodic measure preserving transformation T was constructed on the Lebesgue measure space of the real line $(X, \mathscr{B}, m)$ which accepted a set A with $m(A) = 1$, and such that A was an exhaustive weakly wandering set under the sequence $\mathbb{W} = \{w_i \mid i \geqslant 0\}$ defined as

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follows: Let $w_0=0$, $w_1=2$, and for $i=\varepsilon_0 2^0+\varepsilon_1 2^1+\cdots+\varepsilon_k 2^k$, $\varepsilon_j=0$ or 1 for $0\leqslant j\leqslant k$, let $w_i=\varepsilon_0 2+\varepsilon_1 2^3+\cdots+\varepsilon_k 2^{2k+1}$.

In what follows we show that the transformation T described above is not an isolated example; rather, it belongs to a wide class of ergodic measure preserving transformations defined on an infinite measure space. The transformations we construct, just like the example they generalize, possess a number of interesting properties which were not explicitly stated in [9]. For instance, from the construction of the transformation T it follows that the exhaustive weakly wandering set A behaves quite regularly in filling out the space X. More specifically, for $k=1, 2, \ldots$ let us consider the sequence of integers $n_k=\max\{w\in\mathbb{W}\mid w<2^k\}$ and the sets $D_k=\bigcup_{w\in\mathbb{W},w\leqslant n_k} T^w A$. Then the sets $\{D_k\}$ fill out the space in an orderly manner; namely, $D_k=\bigcup_{0\leqslant i\leqslant n_k} T^i A$. Using standard techniques it is possible to describe within isomorphism ergodic measure preserving transformations defined on an infinite measure space that possess this property. We refer to [3] for details. Further study of these examples reveal number theoretic properties.

In a different direction let us consider the sequence of integers $\mathbb{V}=\{v_i\mid i\geqslant 0\}$ defined by: Let $v_0=0$, $v_1=2^0$, and for $i=\varepsilon_0 2^0+\varepsilon_1 2^1+\cdots+\varepsilon_k 2^k$, $\varepsilon_j=0$ or 1 for $0\leqslant j\leqslant k$, let $v_i=\varepsilon_0 2^0+\varepsilon_1 2^2+\cdots+\varepsilon_k 2^{2k}$. If we consider the left hand point x of the exhaustive weakly wandering set A of this same example [9], then it is not difficult to see that the sequence $\mathbb{V}$ is the return times under T of the point x to A; namely, $\mathbb{V}=\{v\in\mathbb{N}\mid T^v x\in A\}$. Furthermore, it follows that $\mathbb{N}=\mathbb{V}\oplus\mathbb{W}$.

In [1, 11] it is shown that $\mathbb{N}=\mathbb{V}\oplus\mathbb{W}$ for two infinite subsets $\mathbb{V}$ and $\mathbb{W}$ of the set of non-negative integers $\mathbb{N}$ if and only if they have the following form: Let $\{m_i\mid i\geqslant 1\}$ be an infinite sequence of integers with $m_i\geqslant 2$ for all $i\geqslant 1$, and let $M_0=1$ and $M_i=\prod_{1\leqslant j\leqslant i} m_j$ for $i\geqslant 1$. Then $\mathbb{V}=\{v\mid v=\sum x_{2i}M_{2i}\}$ and $\mathbb{W}=\{w\mid w=\sum x_{2i+1}M_{2i+1}\}$, where the sums are finite and are summed over all $i\geqslant 0$ and x_i with $0\leqslant x_i<m_{i+1}$.

The sequences $\mathbb{V}$ and $\mathbb{W}$ described above in connection with the example in [9] have the structure of the construction just described. The sequence $m_i=2$ for all $i\geqslant 1$ is the one that defines the sequences $\mathbb{V}$ and $\mathbb{W}$.

Let us now consider an arbitrary decomposition of $\mathbb{N}=\mathbb{V}\oplus\mathbb{W}$ as the direct sum of two infinite subsets $\mathbb{V}=\{v\}$ and $\mathbb{W}=\{w\}$ of $\mathbb{N}$. We shall construct an ergodic measure preserving transformation T on the Lebesgue measure space of the real line $(X, \mathscr{B}, m)$ with $m(X)=\infty$, for which there exists a set A with $m(A)=1$, and such that A is an exhaustive weakly wandering set for T under the sequence $\mathbb{W}$. The sequence $\mathbb{V}$ shall have a similar interpretation.

We let $\{m_i\mid i\geqslant 1\}$ be the infinite sequence of integers that defines the sequences $\mathbb{V}$ and $\mathbb{W}$ as described above; namely, let $M_0=1$ and $M_i=\prod_{1\leqslant j\leqslant i} m_j$. For $i=0, 1, 2, \ldots$ we let $\alpha_i=m_{2i}$ and $\beta_i=m_{2i+1}$. We then have $\mathbb{V} = \{v \mid v = \sum x_i M_{2i}\}$ and $\mathbb{W} = \{w \mid w = \sum y_i M_{2i+1}\}$, where the sums are finite and are summed over all $i \geqslant 0$ and all x_i, y_i with $0 \leqslant x_i < \beta_i$ and $0 \leqslant y_i < \alpha_i$.

We first use the sequence $\{\alpha_i\}$ to construct a base transformation S on the Lebesgue measure space $(A, \mathscr{B}_0, m_0)$ of the unit interval A with $m(A) = 1$.

For each $i = 1, 2, \ldots$ we consider the measure space $(X_i, \mathscr{B}_i, m_i)$ where $X_i = \{0, 1, 2, \ldots, \alpha_i - 1\}$, $\mathscr{B}_i =$ all subsets of X_i, and $m_i(\eta) = 1/\alpha_i$ for $\eta \in X_i$. Let $(A, \mathscr{B}_0, m_0)$ be the infinite direct product measure space of $(X_i, \mathscr{B}_i, m_i)$ for $i = 1, 2, \ldots$. The set A then consists of all the points $x = (x_1, x_2, \ldots)$ where $0 \leqslant x_i < \alpha_i$ for $i = 1, 2, \ldots$. Next we define the transformation S on A as follows: Suppose $x = (\alpha_1 - 1, \ldots, \alpha_{k-1} - 1, x_k, x_{k+1}, \ldots)$ where $x_k < \alpha_k - 1$ for some $k > 0$ and $x_i = \alpha_i - 1$ for $i < k$, then $Sx = (0, \ldots, 0, x_k + 1, x_{k+1}, \ldots)$. It follows that except for a set of measure zero (namely the dyadic-like rationals in A) S is an ergodic measure preserving transformation defined on the measure space $(A, \mathscr{B}_0, m_0)$ with $m(A) = 1$. The transformation S can be regarded as a generalized adding machine. Next we shall build on top of the measure space $(A, \mathscr{B}_0, m_0)$ a measure space $(X, \mathscr{B}, m)$ and construct a transformation T on it by using the transformation S and the "sky scraper" method.

We define the following subsets of A: for $n = 1, 2, \ldots$ we let $A_n = \{x \in A \mid x = (\alpha_1 - 1, \ldots, \alpha_n - 1, x_{n+1}, \ldots),\ 0 \leqslant x_i < \alpha_i - 1,\ i > n\}$; and $A^n = \{x \in A \mid x = (0, \ldots, 0, x_{n+1}, \ldots),\ 0 \leqslant x_i < \alpha_i - 1,\ i > n\}$. Initially we let $X = A$ and let $B_1^0 = A_1$. For $x \in X - B_1^0$ we define $Tx = Sx$. At this stage T is defined on $X - B_1^0$. Next we let $j_1 = (\beta_1 - 1) M_1$, consider j_1 isomorphic copies of B_1^0, denote them by B_1^i for $1 \leqslant i \leqslant j_1$, and designate by the same letter σ all the isomorphisms $\sigma\colon B_1^i \to B_1^{i+1}$ for $i = 0, 1, \ldots, j_1 - 1$. We attach to X the sets B_1^i for $1 \leqslant i \leqslant j_1$, denote the resulting space by X, and extend T to it as follows: For $x \in B_1^i$, $i = 0, 1, \ldots, j_1 - 1$ we define $Tx = \sigma x$. We let $B_2^0 = T^{k_1} A_2$ where $k_1 = j_1$, and note that $B_2^0 \subset B_1^{j_1}$. For $x \in B_1^{j_1} - B_2^0$ we let $Tx = S(T^{-k_1} x)$. At this stage T is defined on $X - B_2^0$.

Inductively, suppose that T is defined on $X - B_n^0$. We let $j_n = (\beta_n - 1) M_{2n-1}$, consider j_n isomorphic copies of B_n^0, denote them by B_n^i for $1 \leqslant i \leqslant j_n$, and designate by the same letter σ all the isomorphisms $\sigma\colon B_n^i \to B_n^{i+1}$ for $i = 0, 1, \ldots, j_n - 1$. We attach to X the sets B_n^i for $1 \leqslant i \leqslant j_n$, denote the resulting space by X, and extend T to it as follows: For $x \in B_n^i$, $i = 0, 1, \ldots, j_n - 1$ we define $Tx = \sigma x$. We let $B_{n+1}^0 = T^{k_n} A_{n+1}$ where $k_n = j_1 + j_2 + \cdots + j_n$, and note that $B_{n+1}^0 \subset B_n^{j_n}$. For $x \in B_n^{j_n} - B_{n+1}^0$ we let $Tx = S(T^{-k_n} x)$. At this stage T is defined on $X - B_{n+1}^0$.

Continuing this way we obtain the infinite measure space $(X, \mathscr{B}, m)$ and conclude that the transformation T defined on it is ergodic and measure preserving. The transformation T built this way can also be regarded as a transformation constructed by the cutting and stacking method (see [4]).

We proceed to show that the given sequence $\mathbb{W}$ is an exhaustive weakly wandering sequence for the transformation T; the corresponding exhaustive weakly wandering set will be the set A.

We let $W_0^1 = A$. We note that $T^{M_1 - 1}A^1 = B_1^0$, and for $1 \leqslant k < \beta_1$ we let $W_k^1 = \bigcup T^i B_1^0 = \bigcup B_1^i$, where the union is over all i with $kM_1 \leqslant i \leqslant (k+1)M_1 + 1$. Next we observe that $T^{M_1} W_k^1 = W_{k+1}^1$ for $0 \leqslant k < \beta_1 - 1$, and let $W_0^2 = \bigcup_{0 \leqslant k < \beta_1} W_k^1(\text{disj}) = \bigcup_{a \in \mathscr{A}_1} T^a A$ (disj) where $\mathscr{A}_1 = \{a \mid a = \sum x_1 M_1$, where the sums are finite and are summed over x_1 with $0 \leqslant x_1 < \beta_1\}$. More generally, for $n \geqslant 1$ we note that $T^{M_{2n-1} - 1}A^n = B_n^0$, and for $1 \leqslant k < \beta_n$ we let $W_k^n = \bigcup T^i B_n^0 = \bigcup B_n^i$, where the union is over all i with $kM_{2n-1} \leqslant i \leqslant (k+1)M_{2n-1} + 1$. Next we observe that $T^{M_{2n-1}} W_k^n = W_{k+1}^n$ for $0 \leqslant k < \beta_n - 1$, and $W_0^{n+1} = \bigcup_{0 \leqslant k < \beta_n} W_k^n\,(\text{disj}) = \bigcup_{a \in \mathscr{A}_n} T^a A$ (disj), where $\mathscr{A}_n = \{a \mid a = \sum x_i M_{2i-1}$, where the sums are finite and are summed over all i and x_i with $1 \leqslant i \leqslant n$ and $0 \leqslant x_i < \beta_i\}$. It is clear that $W_0^1 \subset W_0^2 \subset \cdots \subset W_0^n \subset \cdots$, and $X = \bigcup_{1 \leqslant n < \infty} W_0^n$. It follows that $X = \bigcup_{a \in \mathbb{W}} T^a A$ (disj).

The above says that the set A with $m(A) = 1$ is an exhaustive weakly wandering set for T under the sequence $\mathbb{W}$. The left hand point $x_0 \in A$, where $x_0 = (0, 0, ..., 0, ...)$, is one of those exceptional points mentioned earlier where the transformations S and T were not defined. However, the forward images of x_0 under both S and T make sense, and if we consider the non-negative integers n such that $T^n x_0 \in A$ then we obtain the sequence $\mathbb{V}$; namely, $\mathbb{V} = \{n \in \mathbb{N} \mid T^n x_0 \in A\}$. We recall that $\mathbb{N} = V \oplus \mathbb{W}$.

More generally, consider two arbitrary subsets $\mathbb{V}$ and $\mathbb{W}$ of the set of non-negative integers $\mathbb{N}$, and following the literature [11] we shall say that $\mathbb{W}$ has a complement $\mathbb{V}$ in $\mathbb{N}$ in case $\mathbb{N} = \mathbb{V} \oplus \mathbb{W}$. From the structure of the sequences $\mathbb{V}$ and $\mathbb{W}$ it is not difficult to see that when a sequence $\mathbb{W}$ has a complement $\mathbb{V}$ in $\mathbb{N}$ then the complement is unique. On the other hand, when the set $\mathbb{N}$ is replaced by the set of all integers $\mathbb{Z}$ then the situation becomes more complicated, and there does not seem to exist a systematic way of constructing complements of $\mathbb{W}$ in $\mathbb{Z}$.

For a sequence of integers $\mathbb{U} = \{u\}$ let us denote by $\mathbb{U} - \mathbb{U} = \{n \in \mathbb{Z} \mid n = u' - u'', u', u'' \in \mathbb{U}\}$. From general properties of the ergodicity of the transformation T it follows that almost all points $x \in A$ visit the set A infinitely often under images of T. Moreover, if we disregard a set of measure zero, then for a point $x \in A$ if we consider the set of integers $\mathbb{U} = \mathbb{U}_x = \{u \in \mathbb{Z} \mid T^u x \in A\}$, then it follows that $T^n A \cap A \neq \varnothing$ for $n \in \mathbb{U} - \mathbb{U}$. The sequence $\mathbb{U} = \mathbb{U}_x$ is the hitting times of the point x to the set A under the transformation T. Next we consider such a point $x \in A$ and let $n \in \mathbb{Z}$ be an arbitrary integer; since A is an exhaustive weakly wandering set for T under the sequence $\mathbb{W}$ we have $T^n x \in T^w A$ or $T^{n-w} x \in A$ for some $w \in \mathbb{W}$. This implies that $n - w = u$ or $n = w + u$ for some $u \in \mathbb{U} = \mathbb{U}_x$. Furthermore, this representation of n as a sum with $w \in \mathbb{W}$ and $u \in \mathbb{U}$ is unique. This follows from the fact that A is a weakly wandering set under $\mathbb{W}$. Namely, since $T^{w'}A \cap T^{w''}A = \varnothing$ for $w', w'' \in \mathbb{W}$ it follows that $T^k\mathbb{W} \cap \mathbb{W} = \varnothing$ for $k \in \mathbb{W} - \mathbb{W}$. It is clear then that $(\mathbb{W} - \mathbb{W}) \cap (\mathbb{U} - \mathbb{U}) = \{0\}$. We summarize the above in the following:

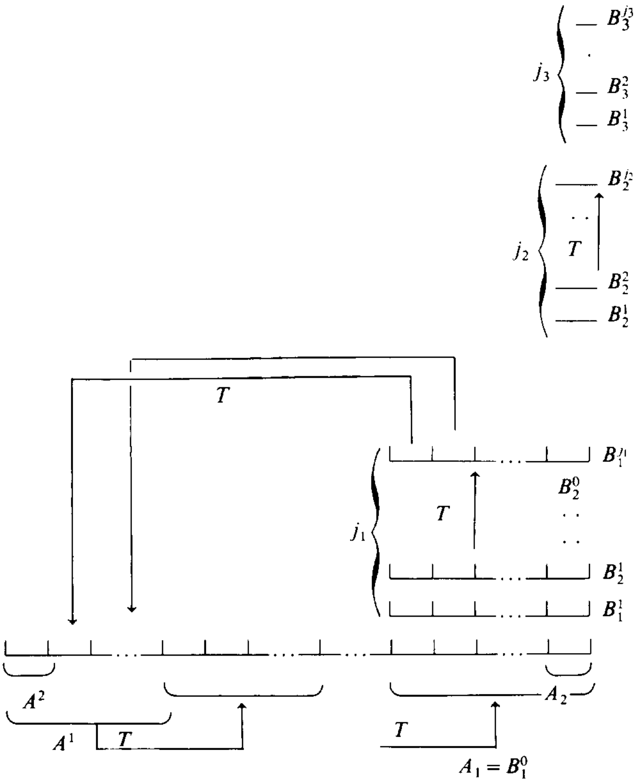
THEOREM. *Let $\mathbb{N} = \mathbb{V} \oplus \mathbb{W}$ be a decomposition of $\mathbb{N}$ into the direct sum of two infinite subsets $\mathbb{V}$ and $\mathbb{W}$, then there exists an ergodic measure preserving transformation T defined on the Lebesgue measure space of the real line $(X, \mathscr{B}, m)$ with $m(X) = \infty$ that satisfies the following*:

(i) *There exists a set $A \in \mathscr{B}$ with $m(A) = 1$ which is an exhaustive weakly wandering set for T under the sequence $\mathbb{W}$.*

(ii) *There exists a point $x \in A$ such that the sequence $\mathbb{V}$ is the return times of the point x to the set A under T; namely, $\mathbb{V} = \{n \in \mathbb{N} \mid T^n x \in A\}$.*

(iii) *Except for a set of measure zero if for a point $x \in A$ we let $\mathbb{U} = \{n \in \mathbb{Z} \mid T^n x \in A\}$, the hitting times of the point x to A under T, then $\mathbb{Z} = \mathbb{W} \oplus \mathbb{U}$.*

We include the following diagram describing the action of the transformation T.



We note that the collection of ergodic measure preserving transformations T we just constructed on the infinite measure space $(X, \mathscr{B}, m)$ possesses the additional property that it admits only measure preserving commutators. In other words, if T' is a measurable transformation with $TT'x = T'Tx$ for almost all $x \in X$ then T' must preserve the same measure m (see also [9]). For ergodic measure preserving transformations defined on a finite measure space this property is an immediate consequence of ergodicity and the fact that the invariant measure is finite. However, this is not a property shared generally by ergodic measure preserving transformations defined on an infinite measure space (see [7]). In our examples this property is a consequence of the fact that there exists an exhaustive weakly wandering set of finite measure. In a joint work with Y. Ito [2] we examine these examples in greater detail and discuss a number of the properties that make them seem to behave like ergodic measure preserving transformations defined on a finite measure space.

In the above theorem part (iii) for certain subsets $\mathbb{W}$ of $\mathbb{N}$ and for almost all $x \in X$ we are able to construct subsets $\mathbb{U} = \mathbb{U}_x$ of $\mathbb{Z}$ that are complements of $\mathbb{W}$ in $\mathbb{Z}$; in other words, $\mathbb{Z} = \mathbb{W} \oplus \mathbb{U}$. We rephrase this fact as a corollary that may be of interest by itself:

COROLLARY. *Let $\mathbb{V}$ and $\mathbb{W}$ be infinite subsets of the set of non-negative integers $\mathbb{N}$ such that $\mathbb{N} = \mathbb{V} \oplus \mathbb{W}$, then there exists a continuum number of subsets $\mathbb{U}$ of $\mathbb{Z}$ such that $\mathbb{Z} = \mathbb{U} \oplus \mathbb{W}$.*

The corollary is an immediate consequence of the theorem; however, all the complements $\mathbb{U}$ in $\mathbb{Z}$ that we obtain by the above method for the subset $\mathbb{W}$ of $\mathbb{N}$ have a very special block pattern. To clarify this slightly, let us consider the sequence $\mathbb{W}$ that was constructed earlier in connection with the example in [9]; namely, $\mathbb{W} = \{$all even powers of 2 and all finite sums of such numbers without repetition$\}$. Then all the complements $\mathbb{U}$ of $\mathbb{W}$ in $\mathbb{Z}$ that we obtain by the above corollary, among other things, have the following property: The set $\mathbb{U} - \mathbb{U}$ contains the integer 1; in fact, the integer 1 occurs infinitely often as a difference of integers in $\mathbb{U}$. This fact is true also for the integers 4, 5, and many others. In a joint work with S. Kalikow [3] we study the above examples in more detail. We show in [3] that for such a sequence $\mathbb{W}$ and the transformation T constructed above it is possible to construct many more and essentially different exhaustive weakly wandering sets. As a consequence, for a given subset $\mathbb{W}$ of $\mathbb{N}$ that has a complement in $\mathbb{N}$ it is possible to enlarge in a significant way the collection of complements of $\mathbb{W}$ in $\mathbb{Z}$ that can be obtained by the above methods. Among other things it is possible to construct complements $\mathbb{U}$ of $\mathbb{W}$ in $\mathbb{Z}$ that do not possess any regularity or block patterns.

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